



# A New Keynesian model with heterogeneous expectations

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## ABSTRACT

Within a New Keynesian model, we incorporate bounded rationality at the individual agent level, and we determine restrictions on expectations operators sufficient to imply aggregate IS and AS relations of the same functional form as those under rationality. This result provides dual implications: the strong nature of the restrictions required to achieve aggregation serve as a caution to researchers—imposing heterogeneous expectations at an aggregate level may be ill-advised; on the other hand, accepting the necessary restrictions provides for tractable analysis of expectations heterogeneity. As an example, we consider a case where a fraction of agents are rational and the remainder are adaptive, and find specifications that are determinate under rationality may possess multiple equilibria in case of expectations heterogeneity.

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## 1. Introduction

During the past decade the new Keynesian monetary model has become a benchmark laboratory for the analysis of monetary policy. Despite a literature that has coordinated on this model, there is a surprisingly diverse set of policy advice: some advocate simple, implementable policy rules (McCallum and Nelson, 1999; Schmitt-Grohe and Uribe, 2005); optimal state-contingent monetary policy (Giannoni and Woodford, 2002; Woodford, 2003; Benigno and Woodford, 2005); or, inflation targeting or inflation forecast targeting (Svensson, 1999; Svensson and Woodford, 2003; Svensson and Williams, 2005). One theme emerging from this literature is that policy should account for its effect on private-sector expectations of future policy actions. Indeed, Woodford (2006) emphasizes the role policy inertia plays in influencing expectations. Even in the presence of non-rational agents, it is still advised that policy be set while remaining cognizant of the feedback onto beliefs, though often with the added condition that the associated rational expectations equilibrium (REE) is stable under learning (Bullard and Mitra, 2002).

The emergence of this theme stems from models that assume a representative agent structure and homogeneous private-sector expectations.<sup>1</sup> This assumption is widespread despite increasing evidence that agents have heterogeneous expectations. For example, Mankiw et al. (2003) document disagreement among both consumers and professional economists in survey data on inflation expectations. Carroll (2003) shows that information slowly diffuses through the

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<sup>1</sup> Heterogeneous expectations between the policymakers and the private sector have been analyzed by Honkapohja and Mitra (2004) and Sargent (1999).

economy. Branch (2004, 2007) finds that survey data on inflation expectations are consistent with a dynamic discrete choice between statistical predictor functions. The dynamic effect of expectations heterogeneity, though, has not yet been studied in the context of micro-founded monetary models.

In this paper, we introduce heterogeneous expectations into a new Keynesian framework. Incorporating heterogeneity into dynamic general equilibrium models may complicate the aggregate dynamics and, in general, requires maintaining the (high dimensional) wealth distribution as a state variable. This paper provides the assumptions sufficient to ensure the laws of motion for the aggregate variables are analytically tractable and easily comparable to those obtained under rationality. In particular, we provide restrictions on the admissible forms of heterogeneous expectations that facilitate decoupling the aggregate dynamics from the dynamics of the wealth distribution.

This paper's principal contribution is the development of aggregate IS and AS relations that are derived from a micro-founded sticky price model and that are also consistent with heterogeneous, and possibly boundedly rational, expectations. Our model's formulation is couched in a yeoman farmer framework along the lines of Woodford (2003). However, instead of assuming that the agents form their expectations rationally, we assume they form beliefs with expectations operators that may not be fully optimal, and we allow for the possibility that expectations operators may differ across groups of agents. The agents' choices are then modeled as being made optimally given their forecasts of the economy's current and future states.

The learning literature has examined a number of boundedly rational expectations operators—for instance, naive and simple adaptive expectations as well as more sophisticated expectations based on least-squares or constant gain learning. In these types of models, expectations of private agents are assumed to satisfy a number of intuitive properties, including linearity and the law of iterated expectations (LIE). Instead of specifying particular expectations operators, we remain agnostic as to the type of agents in our economy and adopt an axiomatic approach: we characterize a class of admissible boundedly rational expectations operators by specifying axioms that they are required to satisfy. These axioms incorporate the intuitive properties mentioned above as well as a few more stringent requirements required to obtain analytic aggregation. These axioms simultaneously allow for a tractable model of heterogeneous beliefs as well as emphasize the explicit and somewhat strong assumptions that underlie the aggregation result.

The resulting reduced form model is similar to the standard IS and AS relations except that conditional expectations are replaced by a general expectations operator that is a convex combination of boundedly rational expectations operators.<sup>2</sup> Also, capturing comparability, we find that if a proportion  $\alpha$  of agents are rational and if  $\alpha \rightarrow 1$  then the reduced form model reduces to the usual new Keynesian IS and AS relations.

The explicit reduced form model in case of heterogeneity in expectations, together with its ability to capture the rational model as a limiting case, allows for the analysis of the dynamic implications of expectations heterogeneity. As an example, we consider the monetary policy implications of our model. It is well known in the new Keynesian monetary policy literature that forward looking policy rules may result in indeterminacy and therefore induce welfare reducing sunspot fluctuations. In Section 3 of our paper, we examine the potential for expectations heterogeneity to exacerbate or mitigate this finding. Following Brock and Hommes (1997), we assume a proportion of agents are rational and the remaining form their expectations adaptively. We find that the impact on indeterminacy of increasing the proportion of adaptive agents depends critically on how the adaptive predictor weighs past data. If past data are discounted, then heterogeneity may be stabilizing in the sense that policy rules that result in indeterminacy under full rationality may yield determinacy when even a small proportion of agents are adaptive. On the other hand, if agents place greater weight on past data, then heterogeneity may be destabilizing in an analogous sense: policy rules yielding determinacy under rational expectations may produce indeterminacy if even a small proportion of agents are adaptive.

The heterogeneous expectations framework facilitates a generalization of the manner in which policy affects private-sector expectations. Importantly, monetary policy influences expectations of current and future policy decisions. The conduct of sound policy depends on the distribution of heterogeneity since heterogeneous expectations can alter the way in which expectations propagate shocks. This reasoning ties in with policymakers' frequently stated priority to prevent inflation expectations from becoming unhinged (e.g. Bernanke, 2004). In the current framework, how aggressive monetary policy needs to be in responding to inflation depends on the distribution of heterogeneity and how strongly boundedly rational agents respond to past data.

This paper is organized as follows. Section 2 presents the axioms that generalizes the new Keynesian model to include heterogeneous expectations. Section 3 discusses in detail the nature of the axioms required for our aggregation result. Section 4 presents an example with an explicit specification of heterogeneity to facilitate analysis of the model's equilibrium properties. Section 5 concludes.

## 2. Heterogeneous bounded rationality

This section develops a version of the New Keynesian model extended to include two types of agents who are identical except in the way they form expectations. We specify expectations axiomatically, in a manner that will still allow for

<sup>2</sup> Gali et al. (2004) develop a model with rational and myopic agents. The heterogeneity is in consumption-saving behavior rather than expectations. In our model, all agents solve the same problem, have the same utility function, and face the same constraints. They differ, however, in how they evaluate their utility flows, which may imply different individual wealth levels.

aggregation across heterogeneous agents. Our primary finding is an aggregation result: we derive log-linearized IS and AS relations that depend on a convex combination of heterogeneous expectations operators; and our axiomatic approach allows us to understand precisely the assumptions needed to obtain aggregation. To ease exposition, we assume a purely forward-looking specification of the model, though it would be straightforward to extend our methodology to a version of the model that incorporates habit persistence and partial price indexation (e.g. [Christiano et al., 2005](#)).

### 2.1. Heterogeneous expectations: axioms

Following [Woodford \(2003\)](#) there is a continuum of private agents indexed by  $i \in [0, 1]$ . These agents are yeoman farmers with linear production functions  $Y^i = N^i$ , where agent  $i$  produces good  $i$  by supplying labor  $N^i$ . Agents are taken to maximize the expected value of discounted utility flow, but instead of assuming that agents form expectations rationally, we allow for more general expectations operators. We assume that a proportion  $\alpha$  of agents forecast future variables using the expectations operator  $E^1$  and the remaining agents use  $E^2$ , and for simplicity, we assume an agent producing good  $i \in [0, \alpha]$  uses  $E^1$  and an agent producing good  $j \in (\alpha, 1]$  uses  $E^2$ . We now develop a framework in which equilibrium inflation and output depend on a heterogeneous expectations operator  $\hat{E}_t$  that is a linear combination of  $E^1, E^2$ :  $\hat{E} = \alpha E^1 + (1 - \alpha)E^2$ .

Our view on expectation formation is influenced, in part, by the adaptive learning literature (e.g. [Marcet and Sargent, 1989](#); [Evans and Honkapohja, 2001](#)). In this literature, fully rational expectations are replaced by linear forecasting rules with parameters that are updated by recursive least squares. We also envision agents engaged in economic forecasting while recognizing there may exist heterogeneity in forecasting rules. Some examples of heterogeneity consistent with our framework include the following: some agents may be rational while others adaptive, as has been examined in a Cobweb model by [Brock and Hommes \(1997\)](#) and found empirically relevant in the data in [Branch \(2004, 2007\)](#); agents may have different information sets (e.g. [Mankiw et al., 2003](#)); or, they may use structurally different learning rules as in [Honkapohja and Mitra \(2005\)](#). Our goal is to extend this notion of agents as forecasters to the agents' primitive problem, and to characterize a set of admissible beliefs that facilitates aggregation into heterogeneous IS and AS relations.

Denote by  $E_t^\tau$  a (subjective) expectations operator; that is,  $E_t^\tau(x_{t+k})$  is the time  $t$  expectation of  $x_{t+k}$  formed by an agent of type  $\tau$ . We impose that

A1. Expectations operators fix observables.

A2. If  $x$  is a variable forecasted by agents and has steady state  $\bar{x}$  then  $E^1 \bar{x} = E^2 \bar{x} = \bar{x}$ .

A3. If  $x, y, x + y$  and  $\alpha x$  are variables forecasted by agents then  $E_t^\tau(x + y) = E_t^\tau(x) + E_t^\tau(y)$  and  $E_t^\tau(\alpha x) = \alpha E_t^\tau(x)$ .

A4. If for all  $k \geq 0$ ,  $x_{t+k}$  and  $\sum_{k \geq 0} \beta^{t+k} x_{t+k}$  are forecasted by agents then

$$E_t^\tau \left( \sum_{k \geq 0} \beta^{t+k} x_{t+k} \right) = \sum_{k \geq 0} \beta^{t+k} E_t^\tau(x_{t+k}).$$

A5.  $E_t^\tau$  satisfies the law of iterated expectations: If  $x$  is a variable forecasted by agents at time  $t$  and time  $t + k$  then  $E_t^\tau \circ E_{t+k}^\tau(x) = E_t^\tau(x)$ .

A6. If  $x$  is a variable forecasted by agents at time  $t$  and time  $t + k$  then  $E_t^\tau E_{t+k}^\tau(x_{t+k}) = E_t^\tau x_{t+k}$ ,  $\tau' \neq \tau$ .

A7. All agents have common expectations on expected differences in limiting wealth.

These assumptions serve several purposes: they impose regularity on the expectations operator consistent with the literature on bounded rationality; they facilitate aggregation; and they emphasize the restrictions needed to obtain aggregation. Below, while deriving the aggregation result, we note where each axiom is employed and illustrate its implications for aggregation. In addition, the following brief discussion of each assumption is useful. Assumption A1 is consistent with reasonable specifications of agent behavior (the forecast of a known quantity should be the known quantity). Assumption A2 requires some continuity in beliefs in the sense that, in a steady state, agents' beliefs will coincide. Assumptions A3 and A4 require expectations to possess some linearity properties. Essentially, linear expectations require agents to incorporate some economic structure into their forecasting model rather than, say, mechanically applying a lag operator to every random variable.

Assumptions A5 and A6 restrict agents' expectations so that they satisfy the LIE at both an individual and aggregate level. The LIE at the individual level is a reasonable and intuitive assumption: agents should not expect to systematically alter their expectations. Assumption A6 is more subtle and is necessary so that aggregate expectations satisfy the LIE. The assumption of LIE at the aggregate level is not without consequence, however: A6 requires that an agent of type  $\tau$ 's expectation of the future expectations of agents of type  $\tau'$  is  $\tau$ 's expectation of that future variable. This places a particular structure on higher-order beliefs. We emphasize that this structure is *necessary* for the aggregate expectations operator to satisfy LIE, and is thus *necessary* for an aggregation result. There is a wide literature that studies the implications of higher-order beliefs: [Townsend \(1983\)](#) finds that incorporating the way in which agents forecast the forecasts of other agents may have interesting dynamic implications; for the incorporation of higher-order beliefs into monetary models, see [Woodford \(2002\)](#), [Amato and Shin \(2006\)](#), and [Kurz \(2008\)](#). Higher-order beliefs may be important in some settings and its consequence for economic dynamics is a topic warrant further study. Below, in Section 3, we illustrate the implications of

relaxing A6 (and A7) and it becomes clear how restrictive these assumptions are in a new Keynesian environment. One important conclusion that can be drawn from this paper is that heterogeneous expectations will alter the dynamic properties of the economy in many important dimensions. These dimension will become evident below.

Assumption A7 (made more precise below) is perhaps the most restrictive assumption. In the formulation of expectations there will be wealth dynamics that differ by expectations type. These wealth dynamics, at first glance, appear to cause a problem for aggregation: if wealth dynamics matter for aggregate variables then it will be necessary to keep track of the wealth distribution and incorporate its evolution into the reduced form model. The axiom that agents agree on limiting wealth distributions avoids this added difficulty thereby allowing us to remain close in form to the homogeneous case. Again, we explore the implications of this assumption in Section 3.

## 2.2. Households

Agents have instantaneous utility functions given by  $u(C_t^i, M_t^i/P_t) - v(N_t^i)$ , where  $U$  is separable and  $C^i, P$  are the CES aggregators

$$C^i = \left( \int C^i(j)^{(\theta-1)/\theta} dj \right)^{\theta/(\theta-1)} \quad \text{and} \quad P = \left( \int P(j)^{1-\theta} dj \right)^{1/(1-\theta)}. \tag{1}$$

Agents choose sequences for consumption, money holdings, labor, and goods prices to maximize expected discounted utility subject to a flow constraint. However, as is standard in the bounded rationality and learning literature, agents do not necessarily observe current values of endogenous variables when making forecasts: it may be that  $E_t^i x_t \neq x_t$ .<sup>3</sup> Quite naturally, we do assume that current values of an individual agent's own choice variables are known.

Under rationality, the agent's contingency plans for the consumption bundle, prices, labor, bonds, and money holdings are chosen to satisfy first order conditions, and these conditions depend in part on conditional expectations formed with respect to the true distributions of the relevant variables. Under bounded rationality, the optimal solution is more subtle. For general subjective expectations  $E_t^i$  it is not obvious that the analogous first order conditions are necessary or sufficient for the contingency plans to be optimal: the theory of dynamic programming and the principle of optimality were developed in stochastic environments assuming  $E_t^i$  are conditional expectations, and in the present setting, it is not clear that the relevant theorems apply.<sup>4</sup> Instead, we follow the learning literature by taking as primitive that agents with subjective (non-rational) expectations choose plans that satisfy the associated Euler equations, and to justify this assumption we appeal to the variational intuition for the usual Euler equations in case of rationality: we assume that agents make choices by equating expected marginal benefit with expected marginal cost.

Agents choose consumption bundles to satisfy an intertemporal Euler equation of the form

$$u_c(C_t^i) = \beta(1 + i_t)E_t^i \left( \frac{P_t}{P_{t+1}} \right) u_c(C_{t+1}^i). \tag{2}$$

Here  $u_c$  is the marginal utility of consumption and  $i_t$  is the nominal interest rate at time  $t$ . Intuitively, the consumption bundle choice must be optimal with respect to the agent's time  $t$  perceived trade-off. Preston (2006) presents an alternative approach in which the agents' optimal plans also respect their perceived lifetime *intertemporal* budget constraint.<sup>5</sup> Explicitly incorporating long-horizon expectations into the model is the next step and is a topic of current research.

We follow Woodford (2003) in assuming the cashless limit case: the utility function is such that each household holds an arbitrarily small amount of real-money so that the central bank can effectively choose a target level for the interest rate and, although the real-money-demand function dictates the size of open market operations consistent with that target, the actual effect of these operations on the net supply of bonds is negligible. In the current environment, the money-demand equation is given by

$$\frac{u_m(C_t^i, M_t^i/P_t)}{u_c(C_t^i, M_t^i/P_t)} = \frac{i_t}{1 + i_t}. \tag{4}$$

Notice as well that the expression for money demand (4) does not depend explicitly on agents' expectations.<sup>6</sup>

<sup>3</sup> This assumption is not necessary for our aggregation result; we make it to remain close to the literature on bounded rationality, and to be consistent with our specification of adaptive expectations in Section 3.

<sup>4</sup> We have not assumed that  $E_t^i$  may be thought of as a mathematical expectation with subjective prior.

<sup>5</sup> As is standard, we do require that the agent's subjective transversality condition, given by

$$\lim_{k \rightarrow \infty} E_t^i \beta^{t+k} u_c(C_{t+k}^i) \frac{B_{t+k}^i}{P_{t+k}} \leq 0, \tag{3}$$

is satisfied ex post, where  $B$  is nominal bond holdings. See Evans et al. (2003) for a discussion.

<sup>6</sup> In incomplete market settings  $i_t$  may depend on the distribution of agents' wealth holdings (e.g. Krusell and Smith, 1998; Aiyagari, 1994, 1995). Our modeling of risk-sharing below makes wealth holdings a fraction of the real-income of their type cohort.

We adopt the Calvo pricing structure, where with a positive probability each farmer's price may remain fixed. Acting as price setters, individual agents face the risk associated with this Calvo lottery.<sup>7</sup> The standard treatment allows agents to hedge against this risk by trading in a complete market for contingent claims; in this way, agents are able to guarantee themselves the average real income each period, and because of this, agents forecast aggregate real income. Our model posits two types of agents. While each type will face an equal probability of being allowed to change prices, in our model, agents may have different expected real income due to heterogeneous expectations. To allow our agents to hedge against the Calvo risk, we assume the presence of a benevolent financial regulator who takes insurance premiums optimally from each type of consumer and then returns the proceeds in order to provide agents the means to hedge the risk associated to the Calvo lottery. The planner collects all income and then redistributes to each type of agent the average income of that agent's type. The planner does this so that each agent type, as indexed by their expectation formation mechanism, is fully insured against the risk associated to the possibility that they will not be tapped to adjust prices; in this way agents are able to guarantee themselves within type average real income. This approach is similar to *Kocherlakota (1996)* except we assume there are no commitment issues, on either side, and this essentially creates two types of representative agents. This risk-sharing mechanism is also employed by *Shi (1999)* and *Mankiw and Reis (2007)* and is entirely standard in models with heterogeneous agents. As an alternative, we could assume that each household consists of a continuum of Yeoman farmers each facing the same Calvo risk, and they share the proceeds across the household. The particulars of the risk sharing mechanism are not consequential to the aggregation result below provided that a risk sharing mechanism exists.

As an alternative there is a long history of studying heterogeneity in dynamic models with incomplete financial markets. In many cases, the existence of heterogeneous consumers and incomplete markets imply that equilibrium prices and allocations differ from the representative agent model, but the quantitative differences seem small (*Krusell and Smith, 1998*). In our setting, heterogeneity arises because agents use different expectations operators when solving for their optimal plans, thus their optimal allocations could be different. Our intent is to provide an axiomatic foundation for the model with heterogeneous expectations to remain close, in reduced form, to the standard model.

Under our financial structure, agents of type  $\tau$  may guarantee themselves a real income  $\Omega^\tau$  as given by

$$\Omega^1 = \frac{1}{\alpha P} \int_0^\alpha P(i)Y(i) di \quad \text{and} \quad \Omega^2 = \frac{1}{(1-\alpha)P} \int_\alpha^1 P(i)Y(i) di. \quad (5)$$

Notice that

$$P(\alpha\Omega^1 + (1-\alpha)\Omega^2) = \int P(i)Y(i) di = PY,$$

from which it follows that  $Y = \alpha\Omega^1 + (1-\alpha)\Omega^2$ .

In equilibrium, agents choices must satisfy their nominal budget constraint:

$$C_t^i P_t + M_t^i + B_t^i + I_{pt}^i = P_t^i Y_t^i + (1 + i_{t-1})B_{t-1}^i + M_{t-1}^i + I_{rt}^i, \quad (6)$$

where  $P^i$  is the price set by agent  $i$ ,  $Y^i$  is the quantity of goods produced by agent  $i$ ,  $M^i$  is the money held by agent  $i$ ,  $B^i$  are the bonds held by agent  $i$ , and  $I_p^i$  and  $I_r^i$  are nominal payments to and receipts from an insurance agency respectively, which the agent takes as given.<sup>8</sup>

If agent  $i$  is of type  $\tau$  then,  $I_{rt}^i = P_t \Omega_t^\tau$  and  $I_{pt}^i = P_t^i Y_t^i$ . Therefore, in equilibrium, the real budget constraint requires

$$C_t^i + b_t^i = \left( \frac{1 + i_{t-1}}{1 + \pi_t} \right) b_{t-1}^i + \Omega_t^\tau, \quad (7)$$

where  $b^i$  is the quantity of real bonds held by agent  $i$ , and  $1 + \pi_t = P_t/P_{t-1}$ , and we impose the cashless limit. A first order expansion of (7) around its steady state, incorporating A2, yields

$$c_t^i - \bar{c}^i + b_t^i = \left( \frac{1 + \bar{i}}{1 + \bar{\pi}} \right) b_{t-1}^i + \Omega_t^\tau - \bar{\Omega}^\tau,$$

where we have exploited  $\bar{b}^i = 0$ . Using  $\beta(1 + \bar{i}) = 1$  and  $\bar{C}^i = \bar{\Omega}^\tau = \bar{Y}$ , we obtain the log-linear approximation

$$c_t^i = \hat{\Omega}_t^\tau \equiv \omega_t^\tau + \beta^{-1} \frac{b_{t-1}^i}{\bar{Y}} - \frac{b_t^i}{\bar{Y}}, \quad (8)$$

where  $c$  and  $\omega$  are in log deviations from steady-state form,  $c^i = \log(C^i/\bar{Y})$ ,  $\omega^\tau = \log(\Omega^\tau/\bar{\Omega}^\tau)$ .

To obtain a new Keynesian IS relation, we log-linearize (2):

$$c_t^i = E_t^i c_{t+1}^i - \sigma^{-1} (i_t - E_t^i \pi_{t+1}). \quad (9)$$

<sup>7</sup> For the purposes of this paper, we abstract from modeling aggregate risk.

<sup>8</sup> Because we assume the case of the cashless limit, we follow *Walsh (2003)* in abstracting from the precise mechanism through which the central bank controls interest rates and injects liquidity into the system.

Inserting (8) into (9) yields the equation

$$\hat{\Omega}_t^\tau = E_t^\tau \hat{\Omega}_{t+1}^\tau - \sigma^{-1}(i_t - E_t^\tau \pi_{t+1}). \tag{10}$$

Note that along any equilibrium path, Eq. (10) must be satisfied for both agent types. This expression incorporates the (log-linearized) Euler equation and budget constraint in a convenient form so that we do not need to explicitly account for bond holdings at the agent level in order to construct an aggregate IS relation. The IS curve comes from aggregating this equation across all households. Iterate this equation forward, while employing assumptions A1, A3, A4, and A5, to obtain

$$\hat{\Omega}_t^\tau = \hat{\Omega}_\infty^\tau - \sigma^{-1} E_t^\tau \sum_{k \geq 0} (i_{t+k} - \pi_{t+k+1}), \tag{11}$$

where  $\hat{\Omega}_\infty^\tau = \lim_{k \rightarrow \infty} E_t^\tau \hat{\Omega}_{t+k}^\tau$ . Noting that bond market clearing requires  $\alpha b_t^1 = -(1 - \alpha)b_t^2$ , we get that

$$\alpha \hat{\Omega}_t^1 + (1 - \alpha) \hat{\Omega}_t^2 = y_t.$$

Then

$$\begin{aligned} y_t &= \alpha \hat{\Omega}_t^1 + (1 - \alpha) \hat{\Omega}_t^2 \\ &= \alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 - \sigma^{-1} \hat{E}_t \sum_{k \geq 0} (i_{t+k} - \pi_{t+k+1}) \\ &= \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1}) + \alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 - \hat{E}_t (\alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2), \end{aligned}$$

where  $\hat{E} = \alpha E_1 + (1 - \alpha)E_2$ . Here we note that the final equality makes use of the LIE at the aggregate level: see assumption A6.

To obtain an IS form most similar to the homogeneous expectations case, we now employ A7, the assumption that all agents predict limiting wealth in an identical manner:

$$\alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 - \hat{E}_t (\alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2) = 0.$$

Assumption A7 can be written equivalently as  $\hat{\Omega}_\infty^j = E^j \hat{\Omega}_\infty^j, j' \neq j$ , that is, agents agree on expected limiting wealth. While this is plausible, and is satisfied when agents have rational expectations regarding limiting wealth or when agents expect to hold steady-state wealth in the long run,<sup>9</sup> one might envision agents holding beliefs that violate it. In that case, the distribution of wealth dynamics will affect aggregate dynamics. This observation provides a nice example of the benefit of our axiomatic approach: Assumption A7 is necessary for our model to aggregate appropriately (see Proposition 1); if A7 does not hold, in practice, then the incorporation of heterogeneous expectations into a new Keynesian model requires considerably more than replacing the rational expectations operator with a convex sum of, say, adaptive expectations operators.

The above analysis provides the following proposition:

**Proposition 1.** *If agents' expectations  $E^1$  and  $E^2$  satisfy assumptions A1–A7 then, up to a log-linear approximation, equilibrium output and inflation satisfy the IS relation*

$$y_t = \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1}) \tag{12}$$

where  $\hat{E} = \alpha E^1 + (1 - \alpha)E^2$ .

From this we conclude that with heterogeneous expectations the IS equation is the usual IS curve where conditional expectations are replaced with a convex combination of the heterogeneous expectations operators.

### 2.3. Optimal pricing

To determine the equilibrium behavior of inflation we turn to the agents' pricing problem. We first compute demand for a particular good: the CES aggregate consumption index implies that demand by an agent of type  $\tau$  for good  $i$ , is given by  $(P_i/P)^{-\theta} (\Omega^\tau + (1 + i_{-1})B_{-1}^\tau/P - B^\tau/P)$ . Aggregating across type and imposing bond market clearing yields

$$Y^i = \alpha (P_i/P)^{-\theta} \Omega^1 + (1 - \alpha) (P_i/P)^{-\theta} \Omega^2 = (P_i/P)^{-\theta} Y$$

as usual.

Note that because of the presence of the insurance company, an agent's income is, in effect, independent of his effort; because of this, an agent's optimal output  $Y^i$  is zero. To avoid this free-rider problem, we assume that private agents are contracted to choose price and output as if they faced their perceived trade-off, and we further assume that these contracts are fully enforceable. In some sense, this requires the agent to behave as if they will receive their full marginal revenue from producing more.<sup>10</sup> With this assumption, we may proceed as follows: Let agent  $i$  be of type  $\tau$ . Let  $C_{t+k}^i = C_{t+k}^i(P_t^i)$  be the

<sup>9</sup> Alternatively, if agents expectations evolve over time, then, on average, agents might expect to all receive the same limiting wealth.

<sup>10</sup> One could equivalently imagine a household where one member consumes given the insurance payment and the other works and makes optimal pricing decisions.



consumption bundle in the event that agent  $i$  cannot change prices for  $k$  periods. Then  $P_t^i$  is chosen by contract to solve

$$\max E_t^\tau \sum_{k \geq 0} (\beta\gamma)^k [u(C_{t+k}^i(P_t^i, \cdot) - v((P_t^i/P_{t+k})^{-\theta} Y_{t+k}))]. \quad (13)$$

Our implementation of bounded rationality requires that agents choose prices to satisfy the first order conditions of this pricing problem, with subjective expectations replacing rational ones. This FOC may be log-linearized as usual, but now there is the presence of both  $Y$  and  $C^i$ . Passing expectations across choice variables (so that  $\hat{\Omega}_t^\tau$  and not  $E_t^\tau \hat{\Omega}_t^\tau$  enters the equation) and imposing equilibrium yields

$$E_t^\tau \sum_{k \geq 0} (\gamma\beta)^k (\log(P_t^i) - \log(P_{t+k}) - \zeta_1 \hat{\Omega}_{t+k}^\tau - \zeta_2 Y_{t+k}) = 0.$$

Subtract  $\log P_t$  from each side and solve for  $\log P_t^i - \log P_t$ . We obtain<sup>11</sup>

$$\log P_t^i - E_t^\tau \log P_t = E_t^\tau \sum_{k \geq 0} (\gamma\beta)^k (\gamma\beta\pi_{t+k+1} + (1 - \gamma\beta)(\zeta_1 \hat{\Omega}_{t+k}^\tau + \zeta_2 Y_{t+k})).$$

Stepping this equation forward, applying A3–A5, and rearranging yields

$$\log P_t^i - \log P_t = \gamma\beta E_t^\tau \pi_{t+1} + (1 - \gamma\beta)(\zeta_1 \hat{\Omega}_t^\tau + \zeta_2 Y_t) + \gamma\beta E_t^\tau \log P_{t+1}^i / P_{t+1}. \quad (14)$$

This equation captures the log-linearization of optimal pricing behavior. While we assume agents' expectations are formed before they observe current aggregate output and price, we allow them to form pricing schedules contingent on current price and output. While this mechanism is not needed for the aggregation result, it does provide an AS relation of precisely the same functional form as that obtained under the assumption of rationality, and is consistent with the learning literature: see [Evans and Honkapohja \(2001\)](#) for a detailed discussion of this mechanism and the related issues.

We turn now to the evolution of aggregate prices. The pricing decision is homogeneous within type, so we now say  $P_t^\tau$  is the optimal price chosen by an agent of type  $\tau$  in time  $t$ , and  $P_t^\tau(j)$  denotes the price set by firm  $j$  (of type  $\tau$ ) in time  $t$ . Then

$$(P_t)^{1-\theta} = \int_0^\alpha P_t^1(j)^{1-\theta} dj + \int_\alpha^1 P_t^2(j)^{1-\theta} dj.$$

The proportion  $1 - \gamma$  of each type changes their optimal price in period  $t$ . Thus

$$\int_0^\alpha P_t^1(j)^{1-\theta} dj = (1 - \gamma)\alpha(P_t^1)^{1-\theta} + \gamma \int_0^\alpha P_{t-1}^1(j)^{1-\theta} dj,$$

and similarly for  $\int_\alpha^1 P_t^2(j)^{1-\theta} dj$ . We obtain

$$(P_t)^{1-\theta} = (1 - \gamma)\alpha(P_t^1)^{1-\theta} + (1 - \gamma)(1 - \alpha)(P_t^2)^{1-\theta} + \gamma(P_{t-1})^{1-\theta}.$$

Log linearization yields

$$p_t = (1 - \gamma)\alpha p_t^1 + (1 - \gamma)(1 - \alpha)p_t^2 + \gamma p_{t-1},$$

so that, subtracting  $(1 - \gamma)p_t$  from both sides, we have

$$\alpha \log P_t^1 / P_t + (1 - \alpha) \log P_t^2 / P_t = \frac{\gamma}{1 - \gamma} \pi_t.$$

Finally, multiply Eq. (14) by  $\alpha$  for  $\tau = 1$  and by  $1 - \alpha$  for  $\tau = 2$  and add to get the following result:

**Proposition 2.** *If agents' expectations  $E^1$  and  $E^2$  satisfy assumptions A1–A7 then, up to a log-linear approximation, equilibrium output and inflation satisfy the AS relation*

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda y_t. \quad (15)$$

### 3. Further implications of heterogeneous expectations

There are many simplifications in the standard New Keynesian model that facilitate representing the equilibrium in terms of reduced-form IS–AS equations. In this paper, we relax one assumption: that expectations are rational. We then ask

<sup>11</sup> This step requires the following observation: Let  $s_t = \sum_{k=1}^t \pi_k = \log P_t - \log P_0$ , and set  $s_0 = 0$ . Then

$$(1 - \delta) \sum_{t \geq 0} \delta^t (\log P_t - \log P_0) = \sum_{t \geq 0} \delta^t s_t - \sum_{t \geq 0} \delta^{t+1} s_t = \sum_{t \geq 0} \delta^t \pi_{t+1}.$$

the following question: In the presence of heterogeneous expectations, what are conditions sufficient to imply that the equilibrium may be represented by analogous IS and AS relations? The benefit of our axiomatic approach is that it makes these conditions clear. The following discussion provides further illustration of the implications of these (potentially) restrictive assumptions.

The aggregation results, captured in Propositions 1 and 2, rely heavily on the LIE, both at the individual level (assumption A5) and at the aggregate level (assumption A6). The LIE at the individual level is a consequence of consistent behavior on the part of agents: private agents should not be able to forecast changes in future forecasts. Incorporation of LIE at the agent level is straightforward: expectations may be constructed recursively by forward iteration on the relevant forecasting models. This is the standard assumption in the literature on learning in which expectations of future variables are based on forward iteration of the agents' perceived laws of motion: see for example, Bullard and Mitra (2002) and Evans and Honkapohja (2003a). Also, it is precisely this construction that we adopt in the next section when imposing adaptive expectations on a proportion of agents.

While we contend that the LIE is a defining property of expectation formation at the individual level, imposing it at the aggregate level is far more restrictive: assumption A6 specifies a particular way in which agents form higher-order beliefs that would not arise under homogeneity. To obtain a tractable aggregation result, one must impose A6; however, if the global interactions under alternate forms of higher-order beliefs are important, then the aggregate dynamics for inflation and output are altered. Similarly, if agents hold differing beliefs about limiting wealth, then aggregate outcomes will depend on the diversity of their beliefs regarding wealth, and the wealth distribution gains added importance for aggregate outcomes.

To illustrate the restrictiveness of Axioms A6 and A7 we consider the implications for the aggregate IS and AS equations in a counterexample where beliefs are so that A6 and A7 are violated. Consider, for example, the derivation of the IS relation without imposing A6 or A7. Following the analysis as in Section 2.2, aggregating across the type-specific Euler equations, and applying A5 yields the following equilibrium condition:

$$y_t = \hat{E}_t y_{t+1} - \sigma^{-1}(i_t - \hat{E}_t \pi_{t+1}) + \alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 - \hat{E}_t(\alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2) \tag{16}$$

$$- \sigma^{-1} \alpha (1 - \alpha) (E_t^1 \circ E_{t+1}^2 - E_t^1 + E_t^2 \circ E_{t+1}^1 - E_t^2) \left( \sum_{k=1}^{\infty} (i_{t+k} - \pi_{t+k+1}) \right) \tag{17}$$

The first line in the RHS expression is the heterogeneous expectations IS relation under assumptions A6 and A7, as presented in Proposition 1. The second line obtains when beliefs are so that A7 is not satisfied. This term, therefore, gives the potential effect on output of deviations in expected limiting wealth. The final line will be non-zero whenever A6 does not hold. This expression, which depends on forecasting the forecasts of the other types, illustrates the impact on aggregate output of higher-order beliefs structures that do not satisfy A6.

This IS equation illustrates that the restrictive assumptions A6 and A7 may have important implications for the determination of output and inflation. Conversely, expressions (16) and (17) show that these assumptions are necessary to obtain the simple heterogeneous expectations IS relation presented in Proposition 1.<sup>12</sup> This analysis also highlights and provides a metric for the dangers of ignoring micro-foundations when contemplating heterogeneous expectations: in case assumptions A6 and/or A7 are thought to be too restrictive, Eqs. (16) and (17) may be viewed as capturing the errors modelers make by imposing a convex combination of expectations operators into the New Keynesian model's reduced form equations.<sup>13</sup>

If, as a counterexample, one agent type fails to satisfy A6 then we can see how a diversity of beliefs will manifest in aggregate outcomes. Suppose that type 2 agents satisfy A1–A7, and type 1 agents satisfy A1–A5 and A7, but not A6. Then the term (17) above becomes

$$- \sigma^{-1} \alpha (1 - \alpha) \left[ E_t^1 E_{t+1}^2 \sum_{k \geq 1} (i_{t+k} - \pi_{t+k+1}) - E_t^1 \sum_{k \geq 1} (i_{t+k} - \pi_{t+k+1}) \right].$$

When A6 fails to hold for one type of agent, then the diversity of beliefs about future aggregate variables will have an even greater impact. Essentially, when the LIE does not hold at the aggregate level, in this case because of agents of type 1, then the heterogeneity of expectations has an extra propagation through these higher-order beliefs. This will make the aggregate output and inflation outcomes differ even further from the representative agent model. It is worth remarking that this insight is similar to Kurz (2008), who argues that aggregate forecasts of future aggregate forecasts (e.g. Axiom A6) may have important implications for macroeconomic outcomes. Macroeconomic models where beliefs do not satisfy Axiom A6 are the topic of current research.

<sup>12</sup> The analogous statement, but without reference to assumption A7, holds for the AS relation obtained in Proposition 2.

<sup>13</sup> If assumptions A6 and/or A7 are violated, the modeler is forced to take a stand on the form of these error terms. For example, if A6 does not hold, then to correctly impose heterogeneous expectations into the IS relation, the modeler must make explicit the computation of

$$E_t^\tau \circ E_{t+1}^\tau (i_{t+k} - \pi_{t+k+1})$$

for all  $t$ , for  $\tau \neq \tau'$ , and for all  $k \geq 1$ . These and related issues are the subject of current research.



By imposing restrictions on higher-order beliefs and expectations of limiting wealth, we are able to derive IS and AS relations that have the same functional form as the IS and AS relations under the assumption of homogeneous rationality. This derivation provides two complementary insights: first, if assumptions A1–A7 are viewed as reasonable, then the implications of expectations heterogeneity may be examined by simply replacing the usual expectations operator with a convex combination of operators; second, and conversely, if assumptions A1–A7 are viewed as unreasonable, then our results suggest that analyzing expectations heterogeneity may be quite complicated and require incorporating wealth into the state variable, and, in particular, studies based on simply replacing the usual expectations operator with a convex combination of, say, adaptive expectations operators, may be misleading.

#### 4. An application to monetary policy

In this section we investigate the aggregate model's determinacy properties. We anticipate that there are other applications of our framework but focus on a particular example that demonstrates the potential for heterogeneity to alter the dynamic properties of the economy.

A linear rational expectations model is said to be determinate if there is a unique non-explosive equilibrium, indeterminate if there are many non-explosive equilibria, and explosive otherwise, and we simply extend these definitions to the model with heterogeneous expectations.<sup>14</sup> That New Keynesian models, when closed with Taylor-type instrument rules, may exhibit indeterminacy has been noted by a number of authors: see, for example, [Bernanke and Woodford \(1997\)](#), [Woodford \(1999\)](#) and [Svensson and Woodford \(2003\)](#). Indeterminacy may be undesirable because of the presence of multiple equilibria, many of which may be welfare reducing. [Clarida et al. \(2000\)](#) have suggested that the volatile nature of the US time-series in the 1970s can, in part, be explained by the presence of sunspot equilibria. Further empirical evidence is provided by [Lubik and Schorfheide \(2004\)](#).

In a univariate reduced form model having the same structure as the heterogeneous NK model—see (23) below—[Branch and McGough \(2005\)](#) found that the presence of adaptive agents impacts the determinacy properties of the model. For some parameterizations, the presence of adaptive agents may be stabilizing in the sense that a model that is indeterminate under the assumption of uniform rationality may be determinate when some agents form their expectations adaptively. The converse also holds: for some parameterizations even if only a very small proportion of agents are adaptive, an otherwise determinate model may be indeterminate.

The importance from a policy perspective of understanding a monetary model's determinacy properties, together with the possibility demonstrated by [Branch and McGough \(2005\)](#) that the presence of adaptive agents can impact determinacy, motivates our analysis of the determinacy properties of the New Keynesian model with heterogeneous expectations. The presence of heterogeneous agents raises an obvious question: how are the determinacy properties of a such a model computed? This question was answered in [Branch and McGough \(2005\)](#): specifically, given a monetary policy  $\{i_t\}$ , and a pair of expectations operators ( $E^1, E^2$ ), solutions to (12), (15) are what [Branch and McGough \(2005\)](#) define as *heterogeneous expectations equilibria*. In this earlier paper, we showed that any model with heterogeneous expectations of a particular form can be re-written in terms of an associated rational expectations model, bringing to bear the standard rational expectations toolkit to analyze the number and nature of heterogeneous expectations equilibria. In this way, the analysis of determinacy in our model reduces to the well-understood determinacy analysis of the associated rational expectations model.

Before analyzing determinacy, we close the heterogeneous expectations New Keynesian model by specifying a monetary policy rule and a precise form for private agents' expectations.

##### 4.1. Monetary policy

The central bank is assumed to use a forward-looking Taylor rule to set interest rates:

$$\text{PR}_1 : \quad i_t = \chi_y E_t y_{t+1} + \chi_\pi E_t \pi_{t+1}, \quad (18)$$

where expectations here are taken to be rational. This expectations based policy rule has been studied by [Bullard and Mitra \(2002\)](#), [Evans and Honkapohja \(2003a,b\)](#), [Evans and McGough \(2005\)](#), and [Preston \(2005\)](#). It has the advantage of conditioning on observables, and thus addresses the concern of [McCallum](#) that contemporaneous policy rules are not realistic in practice; and it imposes that policy-makers set their instrument conditional on their best forecasts of future inflation and output gap (e.g. [Svensson and Woodford, 2003](#)): in this way, policy makers can account for the lag between a policy's implementation and its impact.

We adopt (18) as our benchmark policy rule; however, we also consider the following specification of the policy rule:

$$\text{PR}_2 : \quad i_t = \chi_y \hat{E}_t y_{t+1} + \chi_\pi \hat{E}_t \pi_{t+1}. \quad (19)$$

Here, policy makers react to private agents' expectations rather than to optimal forecasts; this is consistent with the view that policy-makers set policy to anchor agents' expectations. In particular, [Bernanke \(2004\)](#) advocates policy that reacts

<sup>14</sup> Non-explosive equilibria are the focus of rational expectations model because explosive time series typically violate transversality.

aggressively to private-sector expectations, particularly when they are non-rational. Evans and Honkapohja (2003a) show that such a rule may have good properties in the presence of boundedly rational agents. We also checked for robustness when monetary policy adopts rules that depend on lagged and contemporaneous data. The qualitative results presented below are robust to the particular form of the policy rule.

#### 4.2. Rational versus adaptive expectations

In order to present results concerning our model's determinacy properties, we now make specific assumptions on  $\hat{E}_t$ . For simplicity, we focus on rational versus simple adaptive expectations. More specifically, we make the following assumption on beliefs: agents of type 1 are "rational" in that they have one step ahead perfect foresight on output and inflation, and agents of type 2 are adaptive on output and inflation: thus for  $x = y$  or  $\pi$ ,

$$E_t^2(x_t) = \theta x_{t-1},$$

and, as is standard in the learning literature,  $E_t^2 x_{t+k}$  is constructed iteratively. Having defined adaptive expectations on  $x_{t+k}$ , we use assumptions A1–A7 to determine forecasts of other variables. Note that from our construction, adaptive expectations satisfies the LIE. However, somewhat more subtly, rational agents are not fully "rational" in the following sense: full rationality would imply that  $E_t^1 E_{t+1}^2 x_{t+1} = \theta x_t$ ; however, assumption A6 requires that  $E_t^1 E_{t+1}^2 x_{t+1} = E_t x_{t+1}$ . Essentially, we think of the rational agents as really good forecasters, but not fully rational in the conventional sense. More carefully, our rational agents are assumed to know the conditional distributions of output and inflation, but not the expectations of non-rational agents. We further assume that rational agents are not sophisticated enough to back out the expectations of non-rational agents. If the rational agents were fully rational then axiom A6 would not hold and, as demonstrated in Section 3, the aggregate outcomes would be distinct from what is presented below.

Using A6 we obtain

$$\hat{E}_t(x_{t+1}) = \alpha E_t x_{t+1} + (1 - \alpha)\theta^2 x_{t-1}.$$

Here  $\alpha$  is the proportion of agents having rational expectations. The operator  $E_t^2$  is a form of adaptive ( $\theta < 1$ ) or extrapolative ( $\theta > 1$ ) expectations (for simplicity, below we refer to  $E_t^2$  as an "adaptive" expectations operator, even in the event  $\theta > 1$ ).<sup>15</sup> Such expectations can be thought of as arising from a simple linear perceived law of motion  $x_t = \theta x_{t-1}$ .<sup>16</sup> Under certain conditions, including the assumption of homogeneous expectations, it is possible for real-time estimates of  $\theta$  to converge to their REE value. In this paper, we take the value of  $\theta$  as given and leave to future research the study of stability under learning of heterogeneous expectations equilibria.<sup>17</sup>

Adaptive expectations of this form have been considered by Brock and Hommes (1997, 1998), Branch (2002), Branch and McGough (2005), and Pesaran (1987). When  $\theta = 1$  then the operator is usually called "naive" expectations,  $\theta < 1$  are adaptive in the sense that they dampen recent observations,  $\theta > 1$  are often called "extrapolative" or trend chasing expectations. The  $\theta > 1$  is given particular emphasis by Brock and Hommes (1998).

Despite the simple form of our heterogeneous expectations operator, there is evidence in survey data on inflation expectations that agents are distributed across rational expectations and adaptive expectations of this form (see Branch, 2004). The contribution of this paper, however, is to show how to incorporate heterogeneous expectations into a monetary model and show that their presence can have significant implications for monetary policy. Future research should undertake serious empirical studies of heterogeneous beliefs in monetary DSGE models along the lines of Smets and Wouters (2003), Lubik and Schorfheide (2004), or Milani (2007).

#### 4.3. Determining determinacy

Our fully specified aggregate model is given by

$$y_t = \hat{E}_t y_{t+1} - \sigma^{-1}(i_t - \hat{E}_t \pi_{t+1}), \quad (20)$$

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda y_t, \quad (21)$$

$$i_t = \chi_y E_t y_{t+1} + \chi_\pi E_t \pi_{t+1}, \quad (22)$$

$$\hat{E}_t(y_{t+1}) = \alpha E_t y_{t+1} + (1 - \alpha)\theta^2 y_{t-1},$$

$$\hat{E}_t(\pi_{t+1}) = \alpha E_t \pi_{t+1} + (1 - \alpha)\theta^2 \pi_{t-1},$$

<sup>15</sup> In case  $\theta > 1$ , to avoid adaptive agents expecting to have infinite future wealth, we assume that they expect their extrapolative forecasting model to be valid only in the short run, and thus they have a time-varying adaptive model. Alternatively, we could impose that the adaptation parameter used to forecast wealth be less than the inverse of the discount factor.

<sup>16</sup> Other assumptions on the form of the perceived law of motion for adaptive agents may yield distinct results. For example, Lansing (2006) examines a new Keynesian Phillips curve model where agents perceive the truth to follow a random walk with drift. It is possible that in a similar set-up for the framework developed in this paper that, within a self-confirming equilibrium, the optimal weight in an adaptive predictor may be above one, and provide an equilibrium foundation to the analysis below.

<sup>17</sup> Some progress on this issue has been made by Guse (2005) in a simple univariate framework.

together with the alternative policy rule PR<sub>2</sub>.<sup>18</sup> Simplification yields the following reduced form model:

$$Fx_t = BE_t x_{t+1} + Cx_{t-1}, \quad (23)$$

for appropriate matrices  $F$ ,  $B$ , and  $C$ , where  $\det(B) \neq 0$  and  $x = (y, \pi)'$ . Eq. (23) is the rational expectations model (the ARE) associated with the heterogeneous expectations model. Since the ARE has the same form as a rational expectations model with predetermined variables there is an established toolkit for determining determinacy, thereby making our approach easily accessible to practitioners.<sup>19</sup> Techniques for analyzing the determinacy properties of a linear model are well-known: see for example Blanchard and Khan (1980).<sup>20</sup>

Let

$$M = \begin{pmatrix} B^{-1}F & -B^{-1}C \\ I_2 & 0 \end{pmatrix}. \quad (24)$$

The determinacy properties of the model depend on the magnitude of the eigenvalues of  $M$ , which we write as  $\lambda_i$ , where  $i > j \Rightarrow |\lambda_i| \leq |\lambda_j|$ .<sup>21</sup> Note that in general there are two predetermined variables, so that determinacy obtains when precisely two of the eigenvalues of  $M$  are outside the unit circle: fewer eigenvalues outside the unit circle implies indeterminacy and more implies explosiveness. As we will see in the sequel, the notion of indeterminacy can be refined in a useful way: adopting the terminology of Evans and McGough (2005), we say that if  $|\lambda_1| < 1$  then the model is *order 2 indeterminate* (or, equivalently, *exhibits order 2 indeterminacy*) and if  $|\lambda_1| > 1 > |\lambda_2|$  we say the model is *order 1 indeterminate*.<sup>22</sup> This terminology is a particularly intuitive language for describing the degree to which heterogeneity alters the indeterminacy properties. For example, under order 1 indeterminacy, agents coordinate on a single sunspot variable while under order 2 indeterminacy they coordinate on two. Moving from a determinate model to an order 2 indeterminate model would imply that heterogeneity alters the dynamic properties in a fundamentally different way than moving from a determinate model to an order 1 indeterminate model.

#### 4.4. Numerical analysis of determinacy

Numerical results require assigning values to the reduced form model's parameters, i.e. to  $\lambda$ ,  $\gamma$ ,  $\sigma$ , and  $\beta$ , as well as the proportions of rational agents  $\alpha$  and the values of the adaptation parameter  $\theta$ . These parameters could be estimated, however, such an estimation is beyond the scope of this simple application. Instead, we “derive” values for our parameters using estimates available in the literature; and we test for robustness of our results to varying parameter values whenever possible.

The reduced form parameters of the usual New Keynesian model (i.e., the model under rational expectations) are given by  $\lambda$  and  $\sigma^{-1}$ . These parameters have been estimated by a number of authors including Woodford (1999), Clarida et al. (2000) and McCallum and Nelson (1999);<sup>23</sup> see Table 1. The discount factor  $\beta$  is taken to be 0.99.

Given reduced form parameter values, the model is closed by specifying a policy rule (i.e. setting values for  $\chi_y$  and  $\chi_\pi$ ), and choosing values for  $\theta = (\theta_y, \theta_\pi)'$  and  $n = (n_x, n_\pi)'$ . To perform our analysis, we fix  $\theta$  and then for  $\alpha \in \{1, 0.99, 0.9, 0.7\}$  we analyze the determinacy properties of models characterized by the following policy rules:  $0 < \chi_\pi < 2$ ,  $0 < \chi_y < 2$ . We refer to this region as the benchmark policy space. Finally, in the sequel, we will say that certain numerical results “obtain robustly.” By this, we will mean that they obtain across calibrations.

##### 4.4.1. Adaptive expectations: $\theta \leq 1$

We first consider the case where adaptive agents place less weight on past data: that is, when  $\theta < 1$ . Fig. 1 presents results for the Woodford calibration under the benchmark assumptions, with  $\theta = 0.9$ . The NW panel indicates the outcome under rationality, and is consistent with the results in Evans and McGough (2005). Much of the parameter space corresponds to indeterminacy of both order 1 and order 2, with determinacy prevailing only for large  $\chi_\pi$  and low  $\chi_y$ . Now notice that as  $\alpha$  decreases, the region corresponding to determinacy increases in size (as a proportion of the benchmark policy space). Thus we obtain the following result.

**Result 3.** *If  $\theta \leq 1$  then policy rules corresponding to indeterminacy when  $\alpha = 1$  may yield determinacy when there is even a small proportion of adaptive agents in the economy. In this sense, the presence of adaptive agents may be stabilizing.*

<sup>18</sup> Although our model is constructed so that the IS and AS relations are purely “forward looking,” all qualitative results on indeterminacy are robust to a model that includes inertia in the IS and AS equations.

<sup>19</sup> As mentioned above, in rational expectations models, non-explosiveness is typically required in order to avoid violation of asymptotic first order conditions. Because a portion of our agents are rational and therefore have analogous asymptotic first order conditions, we also focus on non-explosive equilibria.

<sup>20</sup> Because we have imposed perfect foresight for our rational agents, indeterminacy in this model will correspond to the presence of multiple bounded perfect foresight paths.

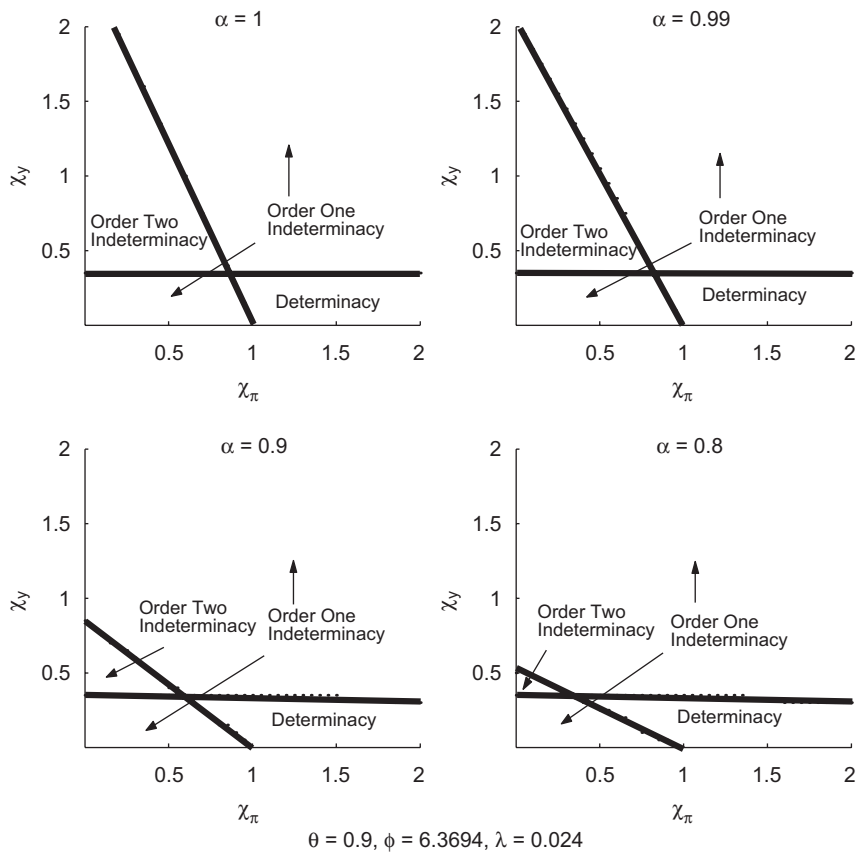
<sup>21</sup> Additive noise in the IS or AS relation does not impact the determinacy properties.

<sup>22</sup> Evans and McGough (2005) study stability under learning in a new Keynesian model with homogenous beliefs.

<sup>23</sup> Calibrations are for quarterly data with  $i_t$  and  $\pi_t$  measured as quarterly rates. The CGG calibration (based on annualized rates) is adjusted accordingly.

**Table 1**  
Calibrations.

Author(s)	$\sigma^{-1}$	$\lambda$
W	1/0.157	0.024
CGG	4	0.075
MN	0.164	0.3



**Fig. 1.** Determinacy properties:  $\theta \leq 1$ .

As indicated in the NW panel, with full rationality, indeterminacy comprises a large portion of the parameter space. To avoid indeterminacy, the Taylor principle instructs policy-makers to “lean against the wind,” by setting the coefficient on inflation (or expected inflation) in the interest rate rule larger than one. In this way, a rise in inflation is met with a larger rise in the nominal rate, and through the Fisher relation this implies a rise in the real interest rate that will have a dampening effect on the economy.

While the Taylor principle does, in many cases, render a model determinate, and while it is wonderfully elegant and wholly intuitive, it is known in general to be neither necessary nor sufficient to guarantee a unique equilibrium: see Bullard and Mitra (2002), Benigno and Woodford (2005), Carlstrom and Fuerst (2005).<sup>24</sup>

Both the Taylor principle, and the fact that it may be neither necessary nor sufficient for determinacy is evident in Fig. 1. The NW panel contains a steep, negatively sloped line anchored to the horizontal axis at  $\chi_\pi = 1$ . For  $\chi_y = 0$ , this line marks the boundary between rules that satisfy the Taylor principle and those that do not. This panel shows that rules which satisfy the Taylor principle, but which also set  $\chi_y > \approx 0.35$  result in order 1 indeterminacy: here the Taylor principle is not sufficient to prevent indeterminacy. It is also evident from the NW panel, that for values of  $\chi_\pi$  which are smaller than but very near to one, there correspond  $\chi_y$  so that the model is determinate, thus demonstrating that the Taylor principle is not necessary for determinacy to obtain.

<sup>24</sup> Benhabib et al. (2001) show that even when it does yield determinacy, a rule satisfying the Taylor principle may be ill-advised.

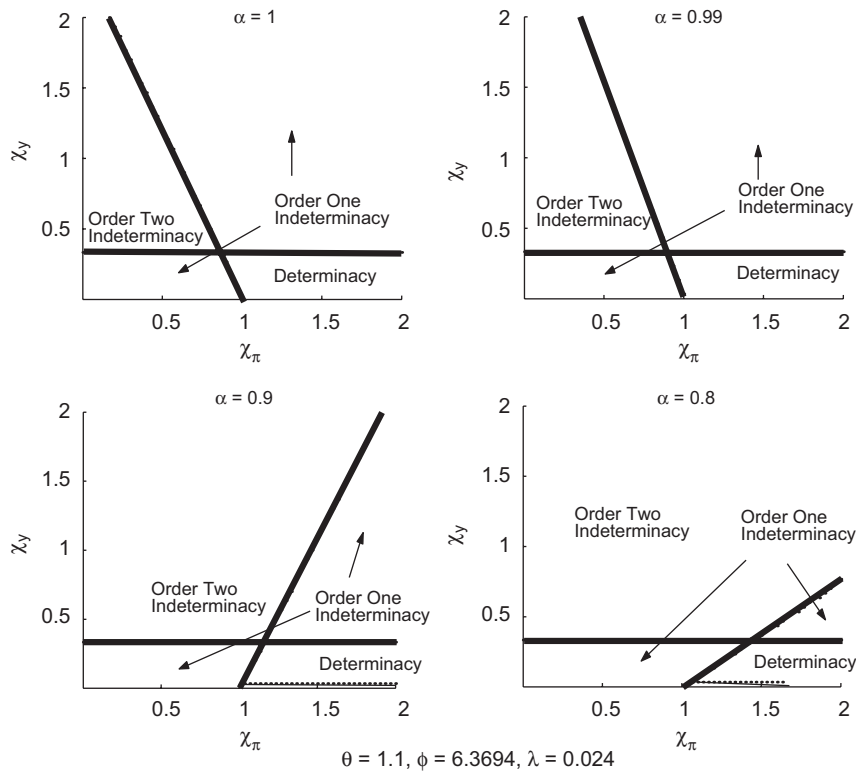


Fig. 2. Determinacy properties:  $\theta > 1$ .

Now notice the impact of heterogeneity: as  $\alpha$  decreases the line anchored at  $\chi_\pi = 1$  rotates counterclockwise, thereby enlarging the relative size of the determinacy region; this rotation therefore increases the number of rules that simultaneously violate the Taylor principle and yield determinacy. We find that this *counterclockwise* rotation of the line anchored at  $\chi_\pi = 1$  obtains robustly.

#### 4.4.2. Extrapolative expectations: $\theta > 1$

We now turn to the case where agents place greater weight on past data, i.e.  $\theta > 1$ . Fig. 2 presents results for the benchmark parameter values. The NW panel is identical to the panel in Fig. 1, but here, as  $\alpha$  decreases, the line anchored at  $\chi_\pi = 1$  rotates *clockwise*, thereby reducing the relative size of the region in parameter space corresponding to determinacy. However, for sufficiently high fractions of adaptive agents (small  $\alpha$ ) the presence of heterogeneity may be stabilizing, in accordance with Result 3. This leads us to the next result.

**Result 4.** *If  $\theta > 1$  then policy rules that yield a determinate outcome in case of rationality may present indeterminacy in case even a very small proportion of agents are behaving adaptively. In this case, we find that the presence of adaptive agents may be destabilizing.*

We find that this clockwise rotation line anchored at  $\chi_\pi = 1$  qualitatively obtains robustly.<sup>25</sup>

Results 3 and 4 suggest that whether heterogeneity stabilizes or destabilizes depends on the distribution of agents across rational and adaptive expectations, and how strongly agents project past data in the adaptive predictor. When agents are adaptive in the more traditional sense, the region of determinacy may be more expansive. However, even slightly extrapolative or trend-chasing agents may destabilize a model that would be determinate under rationality. These results imply that a central bank uncertain about the precise nature of heterogeneity may desire policy robust for all reasonable forms of heterogeneity.

The intuition for these results is clear. The usual “Taylor principle” intuition is that if nominal interest rates are not adjusted more than one for one with expected inflation then aggregate supply shocks (or self-fulfilling shocks to inflation expectations) will be further propagated through lower real rates leading to higher contemporaneous and future inflation. In the heterogeneous expectations case the degree of this future propagation depends on  $\alpha$  and  $\theta$  (i.e., how strongly adaptive agents extrapolate past data). In the case of  $\theta < 1$ , these adaptive beliefs are mean reverting and will not further

<sup>25</sup> In the NE panel, there is a small unlabeled region along the horizontal axis and bounded above by a thin, downward-sloping line. In this region order 2 indeterminacy obtains.

propagate the shock into higher future inflation.<sup>26</sup> Thus, a more tepid response in the nominal interest rate rule is consistent with stabilization. When  $\theta > 1$ , though, adaptive agents are trend-following and so inflation will be placed on a self-fulfilling path unless policy is particularly vigilant against inflation. Notice that this logic is consistent with policymakers' concerns that inflation expectations might become unhinged and out of the Fed's control.<sup>27</sup>

#### 4.5. Discussion

The monetary policy literature, though wide and diverse, typically settles on the same recommendation: set policy so that the REE is determinate. At the heart of this recommendation is the property that a determinate steady state eliminates the potential for multiple equilibria to cause volatility of inflation and output. Our results indicate that a policy rule designed to implement determinacy may lead to inefficient outcomes if there exist heterogeneous expectations.<sup>28</sup>

To illustrate this point in the starkest terms, we parameterized the model so that it is determinate under rational expectations. We showed that with even a very small fraction of agents the determinacy properties may be very different. If agents extrapolate past data, then policy set to ensure a determinate REE may lead to indeterminacy with inefficient inflation and output volatility. Therefore, the results of this paper demonstrate that if policy attempts to achieve a determinate REE in a new Keynesian model and these heterogeneous expectations dynamics are present, the policy-maker may unwittingly destabilize the economy. As such, these results are a cautionary note and suggest that perhaps policy should guard against indeterminacy in heterogeneous expectations models.

Whether heterogeneity is stabilizing or destabilizing depends on the distribution and nature of the heterogeneity. If adaptive agents' expectations are dampening then policy does not have to be quite as vigilant against inflation. Instead, when adaptive agents extrapolate or "trend-chase" then policy needs to be even more aggressive in its stance against inflation. These results are intuitive and seem to align with policymaker concerns such as [Bernanke \(2004\)](#) who emphasizes that adaptive beliefs becoming untethered from policy changes poses a significant challenge. Before representative agent models became standard in macroeconomics, similar findings were common and date to [Kaldor \(1939\)](#). However, there are important open questions such as what specification of heterogeneous expectations is consistent with economic data. The benefit of our axiomatic approach is that it is sufficiently general that it nests many forms of heterogeneity besides rational versus adaptive. However, the motivating example presented in this section demonstrates that heterogeneity can strongly impact a New Keynesian model.

## 5. Conclusion

This paper axiomatized a heterogeneous expectations version of a New Keynesian model from a micro-founded monetary economy with nominal rigidities. The heterogeneous expectations model aggregates into a reduced form whose primary distinction from the representative agent model is that conditional expectations are replaced by a convex combination of expectation operators. Depending upon the beliefs of boundedly rational agents, the new Keynesian aggregate supply relation may depend on expectations future inflation and on current output. As an example illustrating the potential implications of our approach, this paper also examined the impact of expectations heterogeneity on a model's determinacy properties. Our central findings are twofold. Heterogeneity may be stabilizing or destabilizing, depending on the nature of the adaptive expectations mechanism. In case the mechanism is extrapolative, models which are determinate in case of rationality may be indeterminate and hence exhibit sunspot equilibria in the presence of even a small proportion of adaptive agents.

The theory presented here aims to provide greater focus on the assumptions required for a New Keynesian model to incorporate boundedly rational behavior. Our primary theoretical result is to provide the assumptions on beliefs necessary to represent a heterogeneous agent economy with an aggregate law of motion. From an empirical viewpoint, our theoretical analysis is important as recent studies have documented heterogeneous expectations and "disagreement" of beliefs in survey data (e.g. [Carroll, 2003](#); [Mankiw et al., 2003](#); [Branch, 2004](#)). However, how these divergent beliefs might impact reduced form relations governing the evolution of the economy has been an open issue. We illustrate that aggregation is possible provided expectations operators are linear, fix observables, satisfy the LIE, and share common beliefs on differences in limiting wealth.

<sup>26</sup> [Weder \(2004\)](#), in a similar exercise set in a real business cycle model, found that when  $\theta = 1$ , the degree of increasing returns necessary for indeterminacy in creases with the fraction of adaptive agents, a result along the same lines as presented here.

<sup>27</sup> That values of  $\theta \leq 1$  tend to be stabilizing and  $\theta > 1$  tend to indeterminacy and instability suggests a model where there is parameter learning and dynamic predictor selection. In this setting it would be interesting to see if in a calibrated version of the model, whether  $\theta_t$  would tend to a number above 1 pushing the economy into the indeterminacy region. This would then add heterogeneous expectations as a potential explanation of the Great Inflation and Moderation. We leave such an examination to future work.

<sup>28</sup> It has also been emphasized that policy rules should be chosen so that the associated unique equilibrium is stable under learning: see for example, [Bullard and Mitra \(2002\)](#), [Honkapohja and Mitra \(2005\)](#) and [Evans and McGough \(2005\)](#). We note that the rule used to generate the plots in Fig. 8 does produce an equilibrium that is stable under learning.



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