Web Appendix to "Unstable Inflation Targets"*
Appearing in the Journal of Money, Credit and Banking

William A. Branch
University of California, Irvine

George W. Evans
University of Oregon
University of St. Andrews

June 13, 2016

Appendix

Proof to Propositions1-2.

Propositions 1 and 2 provide asymptotic approximations to the learning algorithm

$$\begin{aligned}\theta_t &= \theta_{t-1} + \gamma S_{t-1} X_{t-1} \left(\pi_t - \theta'_{t-1} X_{t-1}\right)' \\ S_t &= S_{t-1} + \gamma \left(X_t X'_t - S_{t-1}\right)\end{aligned}$$

and where $\pi_t = T(\theta_{t-1})' X_{t-1} + \alpha^{-1} r_t$. It is possible to re-write the equations for real-time learning in the form

$$\phi^\gamma_t = \phi^\gamma_{t-1} + \gamma \mathcal{H}(\phi^\gamma_{t-1}, \bar{X}_t)$$

where $\bar{X}_t = (1, \pi_t, \pi_{t-1}, r_t)'$. Verifying many of the technical conditions required for convergence of the learning algorithm is simplified by the fact that the state dynamics, in a neighborhood of the equilibrium of interest, are conditionally linear and can be written as

$$\bar{X}_t \equiv \begin{bmatrix} X_t \\ X_{t-1} \\ r_t \end{bmatrix} = \begin{bmatrix} A(\phi_{t-1}) & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{X}_{t-1} + \begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} W_t$$

where $I, 0$ are conformable matrices, and

$$X_t = A(\phi_{t-1}) X_{t-1} + B W_t$$

Here $X_t = (1, \pi_t)'$ and $W_t = (1, r_t)'$ when dynamics are restricted to a neighborhood of the fundamentals REE or $W_t = (1, \nu_t)'$ when restricted to a set near the deflationary trap. The

*Financial support from National Science Foundation Grant no. SES-1025011 is gratefully acknowledged.

superscript γ highlights the dependence of the parameter estimates on γ. The stochastic approximation approach is to compare the solutions to the continuous time ODE and the discrete time algorithm, and then study the convergence of the continuous time approximating ODE. Thus, define the corresponding continuous time sequence for ϕ_t^γ as $\phi_t^\gamma = \phi_t^\gamma$ if $\tau_t^\gamma \leq \tau < \tau_{t+1}^\gamma$ where $\tau_t^\gamma = \gamma t$.

This Appendix sketches the proof to the propositions by making use of Propositions 7.8 and 7.9 of Evans and Honkapohja, and using arguments in Chapter 14 of Evans and Honkapohja and Branch and Evans (2011). The "mean dynamics" are the solution to the ODE

$$\frac{d\phi}{d\tau} = h(\phi)$$

where $h(\phi) = EH(\phi, \bar{X}_t)$. Notice, in particular, that this is the mean dynamics ODE given in the text:

$$\begin{aligned} \frac{d\theta}{d\tau} &= S^{-1}M(\phi)\left(T(\theta) - \theta\right) \\ \frac{dS}{d\tau} &= M(\phi) - S \end{aligned}$$

where $T(\theta) = (\alpha^{-1}(\alpha - 1)\bar{\pi} + \alpha^{-1}a(1+b), \alpha^{-1}b^2)'$ in a subset of the fundamentals REE and $T(\theta) = (\tilde{\pi}, 0)'$ near the deflationary equilibrium.

Let $\tilde{\phi}(\tau, \phi_0)$ be the solution to the mean dynamics differential equation $\dot{\phi} = h(\phi)$ from an initial condition ϕ_0. Define $U^\gamma(\tau) = \gamma^{-1/2}\left(\phi^\gamma(\tau) - \tilde{\phi}(\tau, \phi_0)\right)$. The two propositions in the text are based on U^γ converging to a Gaussian variable, in a sense made precise below. In particular, for small γ the probability distribution of $U^\gamma(\tau)$ converges to the probability distribution of the solution $U(t)$ to the differential equation

$$dU(\tau) = D_\phi h(\tilde{\phi}(\tau, \phi_0))U(\tau)d\tau + \mathcal{R}^{1/2}(\tilde{\phi}(\tau, \phi_0))dW(\tau)$$

The results below establish that $EU(\tau) = 0$ so that, as $\gamma \to 0, E\phi^\gamma(\tau) = \tilde{\phi}(\tau, \phi_0)$ and $\lim_{\tau\to\infty} \tilde{\phi}(\tau, \phi_0) = \phi^*$. Thus, key properties of the learning dynamics arise from a study of (i.) the asymptotic distribution for θ_t around the rational expectations equilibrium and (ii.) the mean dynamic path $\tilde{\phi}(\tau, \phi_0)$ where ϕ_0 are drawn from the asymptotic distribution.

The validity of the propositions in the text depend on verifying a set of technical conditions. The conditions required for Proposition 2 can be verified by using the arguments in Branch and Evans (2011), and so they are omitted here.

Proposition 2 uses the following result from Evans and Honkapohja (2001):

Proposition 1 (EH(2001)) *Consider the normalized random variables:*

$$U^\gamma(\tau) = \gamma^{-1/2}\left(\phi^\gamma(\tau) - \tilde{\phi}(\tau, \phi_0)\right).$$

As $\gamma \to 0$, the process $U^{\gamma}(\tau)$, $0 \le \tau \le T$, converges weakly to the solution $U(\tau)$ of the stochastic differential equation

$$dU(\tau) = D_{\phi}h(\tilde{\phi}(\tau, \phi_0))U(\tau)d\tau + \mathcal{R}^{1/2}(\tilde{\phi}(\tau, \phi_0))dW(\tau)$$

with initial condition $U(0) = 0$, where $W(\tau)$ is a standard vector Wiener process, and R is a covariance matrix whose i, jth elements are

$$\mathcal{R}^{ij}(\phi) = \sum_{k=-\infty}^{\infty} Cov\left[\mathcal{H}^i(\phi, \bar{X}_k^{\phi}), \mathcal{H}^j(\phi, \bar{X}_0^{\phi})\right]$$

Moreover, the solution to the stochastic differential equation has the following properties

$$\begin{aligned} EU(\tau) &= 0 \\ \frac{dVar(U(\tau))}{d\tau} &= D_{\phi}h(\tilde{\phi}(\tau, \phi_0))V_u(\tau) + V_u D_{\phi}h(\tilde{\phi}(\tau, \phi_0))' + \mathcal{R}(\tilde{\phi}(\tau, \phi_0)), \end{aligned}$$

where $V_u = Var(U(\tau))$. This result indicates that, for finite periods of time, the learning dynamics weakly converge to the solution of the ODE $\dot{\theta} = h(\theta)$, thus establishing Proposition 2.

Proposition 1 relies on the stochastic differential equation in the above result to have a stationary distribution asymptotically. Establishing this result requires stronger conditions. In particular,

A1 ϕ^* is a globally asymptotically stable rest point of the ODE $\dot{\phi} = h(\phi)$.

A2 $D_{\phi}h(\phi)$ is Lipschitz and all of the eigenvalues of $D_{\phi}h(\phi^*)$ have strictly negative real parts.

A3 There exist $q_1, q_2, q_3 \ge 0$ such that, for all $q > 0$ and all compact sets Q, there is a constant $\mu(q, Q)$ such that for all $x \in R^d, a \in Q$,

i. $\sup_n E_{x,a}(1 + |\bar{X}_n|^q) \le \mu(1 + |x|^q)$,

ii. $\sup_n E_{x,a}(|H(\phi_n^{\gamma}, \bar{X}_{n+1})|^2) \le \mu(1 + |x|^{q_1})$,

iii. $\sup_n E_{x,a}(|\nu_{\phi_n^{\gamma}}(\bar{X}_{n+1})|^2) \le \mu(1 + |x|^{q_2})$, where $\nu_{\phi} = \sum_{k \ge 0}(\Pi_{\phi}^k H_{\phi} - h(\phi))(y)$, and Π_{ϕ} is the stationary transition probability associated to the stationary Markov process $\bar{X}_n$,

iiii. $\sup_n E_{x,a}(|\phi_n^{\gamma}|^2) \le \mu(1 + |x|^{q_3})$.

As noted in the text, there are three rest points to the ODE $\dot{\phi} = h(\phi)$, corresponding to the two REE with $b = 0$, i.e. $a = \bar{\pi}$ or $a = \tilde{\pi}$, and the other with $b = \alpha, a = \bar{\pi}$. The $b = \alpha$ REE is unstable under learning, and for some values of ϕ_t^γ the dynamics are explosive. For initial conditions sufficiently close to $b = 0$, and sufficiently small gain parameters γ, then the MSV REE is a stable rest point to the learning dynamics. However, to apply the approximation theorem below, the algorithm needs to rule out trajectories in the explosive region. Thus, the learning algorithm is supplemented with a "projection facility" that projects the iterates ϕ_t^γ into a confined set (see Evans and Honkapohja (2001) and Kushner and Yin (1997)). As a result of these assumptions the RE solution $(\bar{a}, \bar{b}) = (\bar{\pi}, 0)$ is a globally stable rest point of the ODE that satisfies (A1)-(A2).

It remains to verify (A3). Write $\bar{X}_n = \bar{A}(\phi_{n-1})\bar{X}_{n-1} + \bar{B}W_t$, where the expressions for $\bar{A}, \bar{B}$ are given above. The eigenvalues of $\bar{A}$ are zero and A, and the projection facility along with the conditional linearity ensures that $\bar{X}_n$ remains in a compact subset of D, an open set around the REE $(\bar{\pi}, 0)$ or the REE $(\tilde{\pi}, 0)$, which in each case has a unique rest point to $\dot{\phi} = h(\phi)$. Thus (A3.i) is immediate. Verifying conditions (A.ii)-(A.iv) is tedious, but given a projection facility that constrains ϕ_t to lie in a compact subset of D, it is straightforward to extend the arguments in Evans and Honkapohja (2001) (pg.335-336) for the Cobweb model to the present setting.

Proposition 1 arises from the following result in Evans and Honkapohja:

Proposition 2 (EH(2001)) *Consider the normalized random variables* $U^{\gamma_k}(\tau) = \gamma_k^{-1/2}\left(\phi^{\gamma_k}(\tau) - \phi^*\right)$. *For any sequences* $\tau_k \to \infty, \gamma_k \to 0$, *the sequence of random variables* $(U^{\gamma_k}(\tau_k))_{k \geq 0}$ *converges in distribution to a normal random variable with zero mean and covariance matrix*

$$C = \int_0^\infty e^{sB}\mathcal{R}(\theta^*)e^{sB'}ds,$$

where $B = D_\phi h(\phi^*)$.

It follows then that $\theta_t \sim N(\theta^*, \gamma C)$ for small γ and large t. Using arguments in Evans and Honkapohja (2001), Chapter 14.4, C is the solution to the matrix Riccati equation

$$D_\theta h(\phi^*)C + C\left(D_\theta h(\phi^*)\right)' = -\mathcal{R}_\theta(\phi^*)$$

where $R = EH(\phi^*, \bar{X})H(\phi^*, \bar{X})'$. Straightforward calculations then lead to the expression for V in the text.

Overview of the New Keynesian Model with Trend Inflation.

The reduced-form equations (13)-(14) were derived by Ascari and Ropele (2007) from a standard New Keynesian framework and log-linearized around a non-zero steady-state inflation rate. This Appendix provides a brief overview of the model in Ascari and Ropele (2007).

There are a continuum of (identical) households whose flow utility is given by

$$U(C,N) = \frac{C_t^{1-\sigma}}{1-\sigma} - \chi N_t$$

Households maximize lifetime utility subject to the constraint,

$$P_t C_t + B_t \leq P_t w_t N_t + (1 + i_{t-1}) B_{t-1} + \Pi_t + T_t$$

where P_t is the price of the final good, B_t are risk-free one period bonds with nominal net return i_{t-1}, Π_t are profits returned to households and T_t are lump-sum transfers. This formulation assumes the "cashless limit" that abstracts from money balances in the household's problem. The household will select sequences of consumption, labor hours, and bond holdings to satisfy the first-order conditions

$$\begin{aligned} C_t^{-\sigma} &= \beta \hat{E}_t \left(C_{t+1}^{-\sigma} (1+i_t) \frac{P_t}{P_{t+1}} \right) && (1) \\ \chi C_t^{\sigma} &= w_t && (2) \end{aligned}$$

When $\hat{E} = E$, i.e. agents hold rational expectations, the conditions (1)-(2) have the usual interpretation.

When $\hat{E} \neq E$, the equation (1) can be justified in several ways. Our preference is to posit (1) as a behavioral relation that dictates that boundedly rational households choose their consumption to equate their expected marginal rate of substitution with the marginal rate of transformation. This is called Euler-equation learning and is another benchmark approach in the learning literature. See, for example Evans and Honkapohja (2013). In the current setting, the Euler-equation learning interpretation of (1) is completed by imagining a representative agent with a long history of data who observed a strong positive correlation between their own consumption and aggregate consumption. Euler-equation learning is a bounded-optimality approach closely related to the more general "shadow price" learning approach of Evans and McGough (2014).[1]

[1] An alternative approach has been advanced by Preston (2006) in which boundedly rational agents solve their perceived dynamic programming problem, assuming that their beliefs will not change over time. This "anticipated utility" approach is typically implemented, e.g., by obtaining an IS equation that depends on

The final good Y_t is produced by perfectly competitive firms using intermediate goods $Y_t(i)$ produced using a CES production function $Y_t = \left(\int_0^1 Y_t(i)^{(\zeta-1)/\zeta} di\right)^{\zeta/(\zeta-1)}$, $\zeta > 1$. The final goods firms choose their inputs to maximize profits, taking prices as given, resulting in the demand for input i $Y_t(i) = (P_t(i)/P_t)^{-\zeta} Y_t$. Intermediate goods are produced by a continuum of firms with technology $Y_t(i) = N_t(i)$.Intermediate goods producers take the demand for their good as given when setting prices optimally. However, they also face the Calvo risk where with probability α the firm's price will remain unchanged each period. This leads to an expression for price setting that is identical to that of Woodford, except that the optimal re-set price also depends on the cumulative gross inflation rates over the period that a price might remain fixed.

Ascari and Ropele (2007) show that the steady-state properties depend on the trend inflation rate and, in particular, under most plausible parameterizations positive trend inflation leads to a lower steady-state output. Ascari and Ropele then demonstrate that a log-linearization, around a steady-state with gross inflation Π, of the equilibrium conditions lead to the following reduced-form equations:

$$\begin{aligned} \hat{x}_t &= E_t \hat{x}_{t+1} - \sigma^{-1} (i_t - E_t \hat{\pi}_{t+1} - \hat{r}_t) \\ \hat{\pi}_t &= \kappa \hat{x}_t + \beta \Pi E_t \hat{\pi}_{t+1} + (\Pi - 1)\beta(1 - \alpha \Pi^{\zeta-1}) E_t \left((\zeta - 1)\hat{\pi}_{t+1} + \hat{\phi}_{t+1} \right), \\ \hat{\phi}_t &= (1 - \alpha\beta\Pi^{\zeta-1})(1 - \sigma)\hat{x}_t + \alpha\beta\Pi^{\zeta-1} E_t \left((\zeta - 1)\hat{\pi}_{t+1} + \hat{\phi}_{t+1} \right) \end{aligned}$$

where $\hat{x}, \hat{\pi}, \hat{\imath}$ are log deviations from a steady-state with gross inflation factor Π. Iterating forward on the ϕ equation leads to the equations in the text. Ascari and Ropele show that $\kappa = (\Pi - 1)(\sigma - 1)\beta(1 - \alpha\Pi^{\zeta-1}) + \sigma\lambda(\Pi), \lambda(\Pi) = (1 - \alpha\Pi^{\zeta-1})(1 - \alpha\beta\Pi^{\zeta})/\alpha\Pi^{\zeta-1}$.

By setting $\Pi = 1$, i.e. linearizing around a zero inflation steady-state, these equations reduce to the benchmark New Keynesian model

$$\begin{aligned} \hat{x}_t &= E_t \hat{x}_{t+1} - \sigma^{-1} (\hat{\imath}_t - E_t \hat{\pi}_{t+1} - r_t) \\ \hat{\pi}_t &= \beta E_t \hat{\pi}_{t+1} + \kappa \hat{x}_t \end{aligned}$$

References

Ascari, G., and T. Ropele (2007): "Optimal Monetary Policy Under Low Trend Inflation," *Journal of Monetary Economics*, 54, 2568–2583.

expectations of interest rates and inflation over all future horizons. It would also be possible to develop Bayesian approaches that do not rely on a large exogenous component to consumption. However, as is evident from Cogley and Sargent (2008), this approach would be technically difficult to implement. We hypothesize that our results will be robust to these alternative non-RE approaches.

Branch, W. A., and G. W. Evans (2011): "Learning about Risk and Return: A Simple Model of Bubbles and Crashes," *American Economic Journal: Macroeconomics*, 3(3), 159–191.

Cogley, T., and T. J. Sargent (2008): "Anticipated Utility and Rational Expectations as Approximations of Bayesian Decision Making," *International Economic Review*, 49.

Evans, G. W., and S. Honkapohja (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey.

——— (2013): "Learning as a Rational Foundation for Macroeconomics and Finance," in *Rethinking Expectations: The Way Forward for Macroeconomics*, ed. by R. Frydman, and E. S. Phelps. Princeton University Press.

Evans, G. W., and B. McGough (2014): "Learning to Optimize," Working paper.

Kushner, H. J., and G. G. Yin (1997): *Stochastic Approximation Algorithms and Applications*. Springer-Verlag, Berlin.

Preston, B. (2006): "Adaptive Learning, Forecast-based Instrument Rules and Monetary Policy," *Journal of Monetary Economics*, 53, 507–535.